

Second Order Averaging Methods for Oscillatory Control of Underactuated Mechanical Systems

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Abstract: *This paper considers the stabilization of underactuated mechanical systems via high-amplitude, high-frequency actuation. Using higher order averaging techniques, we extend previous work to the case where symmetric products of order higher than one are necessary for controllability. We first introduce a second order averaged mechanical system model that incorporates higher order terms. Using this result, we obtain trajectory tracking in the average by feeding back an error signal that is constant over whole periods of the oscillatory actuation. A simulation demonstrates the method.*

1 Introduction

The use of cyclic action for the task of motion generation is pervasive throughout both nature and the design of mechanical systems. Sideways oscillations of a tail propel fish forward, contractions of muscles in a travelling wave allow a snake to move forward or sideways, and oscillatory inputs allow a car to parallel-park. All of these underactuated mechanical systems use cyclic actuation to produce net motion in directions which are not themselves directly actuated. The control of these and other nonholonomic and underactuated mechanical systems has been widely studied. Prior approaches include discontinuous time-varying control [6], time-varying and averaging methods [16, 12, 14] and hybrid control [2, 18]. Work on stabilization methods based on motion generation with sinusoids is discussed in [19, 15, 5, 9].

Averaging [17] is a useful tool for studying underactuated control systems, as it can address the question of how such systems behave in response to amplitude modulated periodic control signals. Previous work on averaging for control of underactuated systems has focused on systems where accessibility and controllability can be achieved with first level Lie brackets or first level symmetric products [3]. A related work [4] developed higher order expansions but did not apply averaging and is limited to maneuvers with zero final velocity. Both are applied to simple mechanical systems. While many systems of interest meet these requirements, a number

of interesting examples do not satisfy the assumptions of prior methods. Physical examples of such mechanical systems are: the snakeboard [11]; a forced sphere-plate system [7]; and fish-like underwater propulsors [13]. These examples require higher order techniques.

This paper applies higher order averaging methods to mechanical systems that require second level symmetric products or Lie brackets for controllability and accessibility. Sinusoidal actuation at particular frequency combinations has been shown to generate motion in arbitrary directions in the averaged system. Earlier work has achieved stabilization by explicitly determining the time varying amplitudes for the sinusoidal inputs. By simply applying a constant error signal over whole periods of the control signals, we can avoid the complicated results that are inherent in continuous time amplitude modulation. Under our feedback scheme, the system evolves in discrete time, and stability results follow easily. These results are in part an extension of previous work in [3], and a fusion with the work on motion primitives in [4].

Sec. 2 of this paper reviews mechanical systems and their relevant properties. Sec. 3 presents an extension of averaging to include second-order terms. These results are used in Sec. 4 to construct controls that stabilize mechanical system trajectories. We demonstrate these results with a simulation in Sec. 5.

2 Underactuated Mechanical Systems

Recall that the general form of a mechanical system may be written as

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + B(q) = E(q, \dot{q}) + \hat{X}_a(q)u^a(t)$$

where summation over upper and lower indices is assumed, $q \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, M is the mass matrix (or kinetic energy metric), C contains Coriolis and centrifugal terms, B contains potential forces such as gravity, E are applied forces on the system (such as drag), and \hat{X} are the control vector fields. Typically E decomposes into state-feedback terms $E_0(q, \dot{q})$ and dissipative forces $-E_1(q)\dot{q}$. This system can be rewritten as

$$\ddot{q} = M^{-1}(q) \left(-C(q, \dot{q})\dot{q} - B(q) + \tilde{E}(q)\dot{q} + \hat{X}_a(q)u^a(t) \right)$$

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$$= F(q, \dot{q}) + X_0(q, \dot{q}) - D(q)\dot{q} + X_a(q)(1/\epsilon)u^a(t) \quad (1)$$

where the drift term is $F(q, \dot{q}) = -M^{-1}(q)(C(q, \dot{q}) + B(q))$, and the drag term is $D(q)\dot{q} = M^{-1}(q)E_1(q)\dot{q}$. The time-varying vector fields are $X_a(q) = M^{-1}(q)\dot{X}_a(q)$, while the time-invariant state feedback controls are contained in $X_0(q, \dot{q}) = M^{-1}(q)E_0(q, \dot{q})$.

The control vector fields of mechanical systems have useful inherent properties; some important ones being Lie-algebraic. For simple mechanical systems, these Lie-algebraic properties are clarified via the use of homogeneity. As described in Bullo [4], let $x = [q, \dot{q}] \in \mathbb{R}^{2n}$ and define

$$S(x) = \begin{bmatrix} \dot{q} \\ F(q, \dot{q}) + X_0(q, \dot{q}) \end{bmatrix}, \quad X_a^{\text{lift}}(x) = \begin{bmatrix} 0 \\ X_a(q) \end{bmatrix} \\ D^{\text{lift}}(x) = \begin{bmatrix} 0 \\ D(q)\dot{q} \end{bmatrix},$$

so that we have

$$\dot{x} = S(x) - D^{\text{lift}}(x) + X_a^{\text{lift}}(x)u^a(t).$$

The vector fields $S(x)$, $D^{\text{lift}}(x)$, and $X_a^{\text{lift}}(x)$ belong to the set of scalar functions on \mathbb{R}^{2n} which are arbitrary functions of q and homogeneous polynomials in \dot{q}_i of degree 1, 0, and -1 respectively. One can use these homogeneity properties to show that

$$[X_a^{\text{lift}}, X_b^{\text{lift}}] = 0, \quad \langle X_a : X_b \rangle^{\text{lift}} = [X_b^{\text{lift}}, [S, X_a^{\text{lift}}]]$$

where $\langle X_a : X_b \rangle$ is the *symmetric product* between X_b and X_a . These relations lead to further Lie-algebraic equalities which are described below. For details concerning symmetric products see [10] and the references therein.

Henceforth, this paper will make assumptions on the Lie-algebraic properties of the vector fields of Eq. (1) which are a slight generalization of these observations and apply to general mechanical systems, where the vector field components associated with the spray, state feedback, and drag do not exceed degree 1. Because these Lie-algebraic properties hold in the abstract, the equations will be written as

$$\dot{x} = S(x) + Y_0(x) - D(x) + Y_a(x)(1/\epsilon)v^a(t) \quad (2)$$

without regard to the lifted structure of the system, nor the derivation of the spray $S(x)$. The important Lie-algebraic properties of mechanical systems to be utilized are specified below and in equation (14) as assumptions on the above system (2). Assume that the Jacobi-Lie brackets between input control vector fields vanish,

$$[Y_a, Y_b] = 0. \quad (3)$$

Consequently, the Jacobi identity implies that $[Y_a, [S + Y_0 - D, Y_b]] = [Y_b, [S + Y_0 - D, Y_a]]$. Thus, one naturally gains a symmetric product regardless of the control system's inherent homogeneous structure,

$$\langle Y_a : Y_b \rangle \equiv [Y_b, [S + Y_0 - D, Y_a]]. \quad (4)$$

3 Averaging

We would like to examine system response under the application of oscillatory actuation. To begin, rewrite the dynamical system (2) as

$$\dot{x} = f(x) + g(x, t), \quad x(0) = x_0 \quad (5)$$

where $g(x, t)$ is a T -periodic function in t and represents the action of control inputs. We will use the variation of constants formula to analyze this system and its average.

3.1 Variation of Constants

We consider the case of vibrational control, where the inputs are high amplitude, high frequency, i.e.,

$$\dot{x} = f(x) + (1/\epsilon)g(x, t/\epsilon) \quad (6)$$

with ϵ small. First transform time, $t/\epsilon \mapsto \tau$, to obtain

$$\frac{dx}{d\tau} = \epsilon f(x) + g(x, \tau). \quad (7)$$

Now, $f(x)$ is a perturbation to the primary vector field $g(x, \tau)$, and τ is the time variable. Define the following

$$F(y, \tau) = ((\Phi_{0, \tau}^g)^* f)(y) \quad (8)$$

$$\bar{F}(y) = \frac{1}{T} \int_0^T F(y, \tau) d\tau \quad (9)$$

where $\Phi_{0, \tau}^g$ is the flow of the vector field g . According to the variation of constants formula, the solution $x(t)$ is given exactly by

$$x(\tau) = \Phi_{0, \tau}^g(y(\tau)), \quad (10)$$

or as a differential equation by,

$$\dot{x} = g(x, \tau), \quad x(0) = y(\tau) \quad (11)$$

where $\{y(t), t \in [0, T]\}$ is the solution to the system $\dot{y} = \epsilon F(y, \tau)$ with $y(0) = x_0$ as per Eq. (8).

For many problems, it is convenient and suitable to compute an approximate solution that arises from the averaged evolution equation:

$$\dot{z} = \epsilon \bar{F}(z). \quad (12)$$

Using this approach, Bullo [3] derived the following.

Theorem 1 [3] *Let f and g of Eq. (6) be smooth functions in t and x over $\mathbb{R}_+ \times D$. Assume that $z(t)$ of Eq. (12) belongs to the interior of D on time scale 1. Then $x(t) - \Phi_{0, t/\epsilon}^g(z(t)) = O(\epsilon)$ as $\epsilon \rightarrow 0$ on time scale 1.*

Moreover, if $z = 0$ is an asymptotically stable critical point for the linear approximation of \bar{F} , and D is the domain of attraction of $y = 0$, then $x(t) - \Phi_{0, t/\epsilon}^g(y(t)) = O(\delta(\epsilon))$ as $\epsilon \rightarrow 0$ for all t , with $\delta(\epsilon) = o(1)$.

Our goal is to extend Theorem 1 by using higher order averaging to increase the order of approximation. Consequently, the differential equation (12) must be replaced with one based on higher-order averaging. Higher order averaging is required when a system has zero average (whereby higher order terms dominate the dynamics) or when it requires iterated brackets for control.

To compute averaging formulas, the pull-back used in the variation of constants formula must be computed. From Agrachev and Gamkrelidze [1], we have

$$(\Phi_{0,t}^g)^* f = f + \sum_{k=0}^{\infty} \int_0^T \cdots \int_0^{s_{k-1}} (\text{ad}_{g(s_k)} \cdots \text{ad}_{g(s_1)} f) ds_k \cdots ds_1 \quad (13)$$

where the $\{s_j\}$ represent time. The convergence of the infinite sum can be problematic, however if we introduce the following assumption

$$[Y_c, [Y_b, [Y_a, S + Y_0 - D]]] = 0 \quad (14)$$

the sum becomes of finite order. The Lie-algebraic structure of mechanical systems satisfies the above assumption.

3.2 First and Second Order Averaging

Assume that the input functions $v^a(t)$ of Eq. (1) are T -periodic with the following properties,

$$\int_0^T v^a(s_1) ds_1 = 0, \quad \int_0^T \int_0^{s_2} v^a(s_1) ds_1 ds_2 = 0, \quad (15)$$

i.e., the input function is cyclic with zero mean. For convenience, define the autonomous matrix $V = V^{ab}$ by

$$V^{ab} = \frac{1}{2T} \int_0^T (\int_0^{s_1} v^a(s_2) ds_2) (\int_0^{s_1} v^b(s_2) ds_2) ds_1$$

and the time average of a matrix function by

$$\bar{V}(t) = \frac{1}{T} \int_0^T V(t) dt.$$

We will also use the following notation

$$V_{(n)}^{(a)}(t) = \int_0^t \int_0^{s_{n-1}} \cdots \int_0^{s_2} v^a(s_1) ds_1 \cdots ds_{n-1}.$$

For the case where there are multiple upper and lower indices, assume that they are the product of the above type of integral. An example is $V_{(1,1)}^{(a,b)}(t)$ given below.

$$V_{(1,1)}^{(a,b)}(t) = V_{(1)}^{(a)} V_{(1)}^{(b)} = \left(\int_0^t v^a(s_1) ds_1 \right) \left(\int_0^t v^b(s_1) ds_1 \right)$$

Note that $V^{ab} = \frac{1}{2} \bar{V}_{(1,1)}^{(a,b)}(t)$. Additionally define the following

$$\tilde{V}_{(n)}^{(a)} = V_{(n)}^{(a)} - \bar{V}_{(n)}^{(a)}$$

and for the multi-index version

$$\tilde{V}_{(N)}^{(A)} = V_{(N)}^{(A)} - \bar{V}_{(N)}^{(A)}$$

where $(A) = (a_1, a_2, \dots, a_{|A|})$ and $(N) = (n_1, n_2, \dots, n_{|N|})$.

We now modestly extend a theorem of Bullo [3].

Theorem 2 (First order averaging) Consider the system (2) and the initial value problem

$$\dot{z} = S(z) + Y_0(z) - D(z) - V^{ab} \langle Y_a : Y_b \rangle \quad (16)$$

with $z(0) = z_0$ where $z \in \mathbb{R}^{2n}$. Assume the control vector fields and input forcing are smooth functions of their respective arguments and that the Lie bracket properties of (3) and (14) hold. Then $q(t) - \Phi_t^g(z(t)) = O(\epsilon)$ as $\epsilon \rightarrow 0$ on the time scale 1, and $q(t) - \Phi_t^g(z(t)) = O(\delta(\epsilon))$ as $\epsilon \rightarrow 0$ for all t , if $z = 0$ is an asymptotically stable critical point for the linear approximation of the system in (16).

In our version of the theorem, the vector fields in the symmetric product $\langle Y_a : Y_b \rangle$ need not be lifts. The proof of this theorem basically follows from [3].

Our main averaging result given below builds upon the following second order averaging theorem of Sanders and Verhulst [17].

Theorem 3 [17] Suppose that f in Eq. (6) has a Lipschitz continuous first derivative in x on domain D and is continuous in x and t on $D \times \mathbb{R}_+$, and that $y(t)$ belongs to an interior subset of D on the time scale $1/\epsilon$. Then $x(t) - y(t) = O(\epsilon^2)$ as $\epsilon \rightarrow 0$ on the time scale $1/\epsilon$. The function $y(t)$ is given by

$$y(t) = z(t) + \epsilon w(z, t) + O(\epsilon^2)$$

where z is given by

$$\dot{z} = \epsilon \bar{f}(z) + \epsilon^2 \bar{g}(z)$$

and the functions w and g are defined,

$$w(z, t) = \int_0^t (f(z, t) - \bar{f}(z)) dt + a(z) \quad (17)$$

$$g(z, t) = D_z f(z, t) w(z, t) - D_z w(z, t) \bar{f}(z) \quad (18)$$

with $a(z)$ a smooth function making the period time average of w zero.

Note, the time average of $g(z, t)$ may be rewritten as

$$\bar{g}(z) = \frac{1}{2} \left[\int_0^t f(z, \tau) d\tau, f(z, t) \right] + [a(z), \bar{f}(z)]. \quad (19)$$

Our main averaging result follows.

Theorem 4 (Second order averaging) Consider the system (2) and the initial value problem

$$\begin{aligned} \dot{z} = & S(z) + Y_0(z) - D(z) - V^{ab} \langle Y_a : Y_b \rangle \\ & + \frac{1}{2} \epsilon \bar{V}_{(2,1)}^{(a,b)}(t) [[Y_a, S + Y_0 - D], [Y_b, S + Y_0 - D]] \\ & - \frac{1}{2} \epsilon \left(\bar{V}_{(2,1,1)}^{(a,b,c)}(t) - \bar{V}_{(2)}^{(a)} \bar{V}_{(1,1)}^{(b,c)} \right) \cdot \langle Y_a : \langle Y_b : Y_c \rangle \rangle \\ & + \frac{1}{4} \epsilon \left(\frac{1}{2} \int_0^t \bar{V}_{(1,1)}^{(a,b)}(\tau) d\tau \bar{V}_{(1,1)}^{(c,d)}(t) - \int_0^t \tilde{V}_{(1,1)}^{(a,b)}(\tau) d\tau \right. \\ & \left. \bar{V}_{(1,1)}^{(c,d)}(t) \right) \cdot [\langle Y_a : Y_b \rangle, \langle Y_c : Y_d \rangle] \end{aligned} \quad (20)$$

with $z(0) = z_0$. If the control vector fields and input forcing are smooth functions of their respective arguments and the Lie bracket properties of (3) and (14) hold, then $q(t) - \Phi_t^g(z(t)) = O(\epsilon)$ as $\epsilon \rightarrow 0$ on the time scale 1. Furthermore $q(t) - \Phi_t^g(z(t)) = O(\delta(\epsilon))$ as $\epsilon \rightarrow 0$ for all t , if $z = 0$ is an asymptotically stable critical point for the linear approximation of the system in (20).

Proof: We must find an expression for: $\epsilon \bar{F}(y) + \epsilon^2 \bar{G}(y)$. From Theorem 2, $\bar{F}(y)$ is:

$$\bar{F}(y) = S(y) + Y_0(y) - D(y) + V^{ab} \langle Y_a : Y_b \rangle.$$

We analyze $\bar{G}(y)$ in pieces, as per Eq. (19)

$$\bar{G}(y) = \frac{1}{2} \left[\int_0^t F(y, \tau) d\tau, F(y, t) \right] + [A(y), \bar{F}(y)]$$

With $\tilde{F}(y, \tau) = F(y, \tau) - \bar{F}(y)$, the function $W(y, t)$ is defined as

$$W(y, t) = \int_0^t \tilde{F}(y, \tau) d\tau + A(y)$$

where $A(y)$ is determined by requiring that the time-average of $W(y, t)$ vanish. We will need:

$$F(y, t) = S(y) + Y_0(y) - D(y) + V_{(1)}^{(a)}(t) [Y_a, (S + Y_0 - D)] - \frac{1}{2} V_{(1,1)}^{(a,b)} \langle Y_a : Y_b \rangle$$

The expression $\tilde{F} = F(y, t) - \bar{F}(y)$ integrated yields

$$W(y, t) = V_{(2)}^{(a)}(t) [Y_a, (S + Y_0 - D)] - \frac{1}{2} \int_0^t \tilde{V}_{(1,1)}^{(a,b)}(\tau) d\tau \langle Y_a : Y_b \rangle + A(y) \quad (21)$$

whereby $A(y)$ is calculated to be

$$A(y) = -\frac{V_{(2)}^{(a)}(t)}{2} [Y_a, (S - Y_0 - D)] + \frac{1}{2} \int_0^t \tilde{V}_{(1,1)}^{(a,b)}(\tau) d\tau \langle Y_a : Y_b \rangle.$$

The brackets comprising the expression for $\bar{G}(y)$ evaluate as follows.

$$\begin{aligned} & \left[\int_0^t F(y, \tau) d\tau, F(y, t) \right] \\ &= \left(V_{(1)}^{(a)}(t) t - V_{(2)}^{(a)}(t) \right) \cdot \\ & \quad [(S + Y_0 - D), [Y_a, (S + Y_0 - D)]] \\ & + V_{(2,1)}^{(a,b)}(t) [[Y_a, (S + Y_0 - D)], [Y_b, (S + Y_0 - D)]] \\ & - \left(\frac{1}{2} V_{(1,1)}^{(a,b)}(t) t - \frac{1}{2} \int_0^t V_{(1,1)}^{(a,b)}(\tau) d\tau \right) \cdot \\ & \quad [(S + Y_0 - D), \langle Y_a : Y_b \rangle] \\ & - \left(\frac{1}{2} V_{(2)}^{(a)}(t) V_{(1,1)}^{(b,c)}(t) - \frac{1}{2} V_{(1)}^{(a)}(t) \int_0^t V_{(1,1)}^{(b,c)}(\tau) d\tau \right) \cdot \\ & \quad \langle Y_a : \langle Y_b : Y_c \rangle \rangle \\ & + \frac{1}{4} \int_0^t V_{(1,1)}^{(a,b)}(\tau) d\tau V_{(1,1)}^{(c,d)}(t) [\langle Y_a : Y_b \rangle, \langle Y_c : Y_d \rangle]. \end{aligned}$$

Taking the time average and utilizing the integration of

products formula on some of the coefficients,

$$\begin{aligned} & \left[\int_0^t F(y, \tau) d\tau, F(y, t) \right] \\ &= -2 V_{(2)}^{(a)}(t) [(S + Y_0 - D), [Y_a, (S + Y_0 - D)]] \\ & + V_{(2,1)}^{(a,b)}(t) [[Y_a, (S + Y_0 - D)], [Y_b, (S + Y_0 - D)]] \\ & + \int_0^t \tilde{V}_{(1,1)}^{(a,b)}(\tau) d\tau [(S + Y_0 - D), \langle Y_a : Y_b \rangle] \\ & - V_{(2,1,1)}^{(a,b,c)}(t) \langle Y_a : \langle Y_b : Y_c \rangle \rangle \\ & + \frac{1}{4} \int_0^t V_{(1,1)}^{(a,b)}(\tau) d\tau V_{(1,1)}^{(c,d)}(t) [\langle Y_a : Y_b \rangle, \langle Y_c : Y_d \rangle]. \end{aligned}$$

Next, the contribution from the integration constant is,

$$\begin{aligned} & [A(y), \bar{F}(y)] \\ &= V_{(2)}^{(a)}(t) [(S + Y_0 - D), [Y_a, (S + Y_0 - D)]] \\ & + V_{(2)}^{(a)}(t) V^{bc} \langle Y_a : \langle Y_b : Y_c \rangle \rangle \\ & - \frac{1}{2} \int_0^t \tilde{V}_{(1,1)}^{(a,b)}(\tau) d\tau [(S + Y_0 - D), \langle Y_a : Y_b \rangle] \\ & - \frac{1}{4} \int_0^t V_{(1,1)}^{(a,b)}(\tau) d\tau V_{(1,1)}^{(c,d)}(t) [\langle Y_a : Y_b \rangle, \langle Y_c : Y_d \rangle]. \end{aligned}$$

Assembling all of the terms yields:

$$\begin{aligned} \bar{G}(y, t) &= \frac{1}{2} V_{(2,1)}^{(a,b)}(t) [[Y_a, S + Y_0 - D], [Y_b, S + Y_0 - D]] \\ & - \frac{1}{2} \left(\frac{V_{(2,1,1)}^{(a,b,c)}(t)}{2} - \frac{V_{(2)}^{(c)}(t) V_{(1,1)}^{(a,b)}(t)}{2} \right) \langle Y_c : \langle Y_a : Y_b \rangle \rangle \\ & + \frac{1}{4} \left(\frac{1}{2} \int_0^t V_{(1,1)}^{(a,b)}(\tau) d\tau V_{(1,1)}^{(c,d)}(t) - \int_0^t \tilde{V}_{(1,1)}^{(a,b)}(\tau) d\tau \right. \\ & \quad \left. V_{(1,1)}^{(c,d)}(t) \right) [\langle Y_a : Y_b \rangle, \langle Y_c : Y_d \rangle]. \end{aligned}$$

Using the terms computed above for the second order average $dz/d\tau = \epsilon \bar{F}(z) + \epsilon^2 \bar{G}(z)$, a transformation of time back to t and application of Thm. 1 rederived using second-order averaging gives the result. ■

4 Trajectory Stabilization

Given a configuration controllable system of order 2 (see [10] for a discussion of configuration controllability) having the form (2) and the Lie bracket properties (3) and (14), we would like to choose appropriate oscillatory feedback controls to either stabilize the system or track a trajectory. To this end, we will apply Thm. 4 using controls with amplitudes generated by a discretized system error signal and show that a linearization of the result is stable under appropriate choice of gain constants.

If the system (2) is configuration controllable of order 2, we know that there exists a set of linearly independent vector fields $Y_a, \langle Y_a : Y_b \rangle, \langle Y_a : \langle Y_b : Y_c \rangle \rangle$ that span \mathbb{R}^n . In what follows, we give a procedure to construct the required controls (see [14]). For the elements $Y_{ab} = \langle Y_a : Y_b \rangle$ from this set define

$$\zeta_{ab}^a = \alpha_{ab} \lambda_{ab} \sin(\lambda_{ab} t), \quad \zeta_{ab}^b = -\lambda_{ab} \sin(\lambda_{ab} t) \quad (22)$$

and for the elements $Y_{abc} = \langle Y_a : \langle Y_b : Y_c \rangle \rangle$ define

$$\begin{aligned} \xi_{abc}^a &= -\mu_{abc} \cos(\mu_{abc} t), \quad \xi_{abc}^b = -\mu_{abc} \cos(\mu_{abc} t) \\ \xi_{abc}^c &= \beta_{abc} \mu_{abc} \cos(2\mu_{abc} t) \end{aligned} \quad (23)$$

where $\lambda_{ab}, \mu_{abc} \in \mathbb{Z}^+$ and the α_{ab}, β_{abc} are scalar constants. Define a lexicographical ordering on the pairs ab and triples abc such that $ab < cd$ if $a \leq c$ and $b < d$ and similarly for abc . Then choose the frequencies $\lambda_i = \lambda_{i-1} + 1$, $\mu_1 = \lambda_{m-1, m} + 1$, and $\mu_i = 2\mu_{i-1} + 1$. Now sum the appropriate components for each vector field to get the control functions

$$u^a(t) = \sum_{ij} \zeta_{ij}^a + \sum_{ijk} \xi_{ijk}^a \quad (24)$$

By direct computation one can check that

$$\overline{V_{(2,1)}^{(a,b)}} \equiv 0, \quad \overline{V_{(2)}^{(a)}} \overline{V_{(1,1)}^{(b,c)}} \equiv 0$$

Also, note that for mechanical systems the last term in the summation of (20) is identically zero via their Lie-algebraic properties. The averaged system will then have the form

$$\dot{z} = S(z) + Y_0(z) - D(z) + B(z)H(\alpha, \beta) \quad (25)$$

where

$$B(z) = [\langle Y_a : Y_b \rangle, \langle Y_a : \langle Y_b : Y_c \rangle \rangle]$$

$$H(\alpha, \beta) = \begin{bmatrix} -\frac{1}{2} \overline{V_{(1,1)}^{(a,b)}} \\ -\frac{1}{2} \epsilon \overline{V_{(2,1,1)}^{(a,b,c)}} \end{bmatrix}.$$

Theorem 5 Consider a mechanical system of the form (2), which is configuration controllable with first and second level symmetric products, and where the dimensions of the spaces spanned by Y_a , $\langle Y_a : Y_b \rangle$, and $\langle Y_a : \langle Y_b : Y_c \rangle \rangle$ are respectively, m , n_{ab} , and n_{abc} . Assume that there exist functions of the form (22) and (23) such that the linearization of $H(\alpha, \beta)$ with respect to α and β is invertible on the subspace to control, and let $z(t)$ be the averaged system response. Then there exists $K \in \mathbb{R}^{(n_{ab}+n_{abc}) \times 2n}$ such that for

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = -Kz(T[t/T])$$

where $\alpha \in \mathbb{R}^{n_{ab}}$, $\beta \in \mathbb{R}^{n_{abc}}$, we have the stabilized average system response $\lim_{t \rightarrow \infty} z(t) = 0$.

Proof: Given the assumptions on the system, the averaged system (25) is controllable. Linearizing the system with respect to z , α and β yields

$$\dot{z} = Az + B \frac{\partial H}{\partial [\alpha, \beta]} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = Az + B\Gamma \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \quad (26)$$

For the averaging result to hold, the system parameters α and β must be constant over whole periods. Allowing the values of the parameters to be modified at the endpoints of each whole period results in a discrete time

system, the dynamics are obtained by direct integration of (26):

$$z(T) = e^{AT} z(0) + e^{AT} \int_0^T e^{-A\tau} d\tau B\Gamma \begin{bmatrix} \alpha \\ \beta \end{bmatrix}. \quad (27)$$

To perform this computation, note that the matrix A can always be block diagonalized by a state space transformation such that the real part of the eigenvalues of the upper left block (of dimension $2(n-m) \times 2(n-m)$) correspond to the states that are directly controlled and in the average are all negative, those of the middle block are either positive or negative and those of the lower right block are all zero. For now we will assume that all eigenvalues are strictly real. The extension to the general case is straightforward. We will assume that A in (27) is in this block diagonal structure.

Now we effectively have the discrete, linear system

$$z(h+1) = \hat{A}z(h) + \hat{B} \begin{bmatrix} \alpha(h) \\ \beta(h) \end{bmatrix}$$

where $\hat{A} = \text{diag}[e^{A_1 T}, e^{A_2 T}, I + A_3 T]$ and

$$\hat{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & e^{A_2 T} A_2^{-1} (I - e^{-A_2 T}) & 0 \\ 0 & 0 & TI + \frac{1}{2} T^2 A_3 \end{bmatrix} B\Gamma$$

The proof is completed by finding any matrix K such that with

$$\begin{bmatrix} \alpha(h) \\ \beta(h) \end{bmatrix} = Kz(h),$$

the eigenvalues of $\hat{A} - \hat{B}K$ are within the unit circle. ■

Comments. This theorem stabilizes an equilibrium point of our averaged system. To track a trajectory, we simply replace $q(t)$ and $\dot{q}(t)$ with $q(t) - q_d(t)$ and $\dot{q}(t) - \dot{q}_d(t)$. The original system will, in general, oscillate about this equilibrium point (see Guckenheimer and Holmes [8] for the preservation of stability). Due to the periodic nature of the controls, the Nyquist rate is a limiting factor in tracking a trajectory.

5 Example

To demonstrate the preceding theory, we consider a simple non-physical example. It is a second order system having only the following control inputs

$$Y_1(q) = [1, 0, q_2, 0, q_2^2], \quad Y_2(q) = [0, 1, 0, q_1^2, 0]. \quad (28)$$

The drift term integrates velocities, drag is nonexistent, and the state-feedback with $\nu_i = 3q_i + 4\dot{q}_i$, is:

$$Y_0(q, \dot{q}) = [-\nu_1, -\nu_2, -q_2\nu_1, -q_1^2\nu_2, -q_2^2\nu_1].$$

The time-varying control inputs for the symmetric products $\langle Y_1 : Y_2 \rangle$, $\langle Y_1 : \langle Y_2 : Y_1 \rangle \rangle$, and

$\langle Y_2 : \langle Y_1 : Y_2 \rangle \rangle$ follow the earlier construction, with $\lambda_{12} = 1$, $\mu_{112} = 3$, $\mu_{212} = 7$, and $\epsilon = 1/7$. The nonzero gains, $[0.45, 1.75, 18, 60, 15, 80]$, corresponding in pairs to the above symmetric products, then satisfy the requirements of Thm. 5. In Fig. 1, the control system is applied to a ramp input, where the slopes for the individual states are $[\frac{\epsilon^2}{2}, \frac{\epsilon^2}{2}, \frac{\epsilon^2}{2}, \frac{1}{2}, \frac{1}{2}]$. Theorem 5 dictates that the system is stable in the average, corresponding to a stable periodic orbit of size proportional to ϵ , about the trajectory for the actual system. The slopes differ by a factor of ϵ according to the order with which they are controlled, i.e., directly (ϵ^2), with first level symmetric products (ϵ), or with second level symmetric products (ϵ^0). The factors are required so that the nonlinear terms of the directly controlled states do not dominate over the controls of the indirectly controlled states.

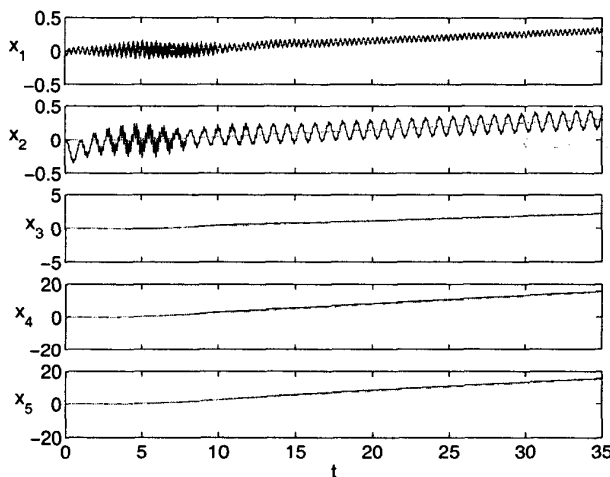


Figure 1: Trajectory tracking results for example.

6 Conclusions and Future Work

This paper showed how to extend earlier work on averaging for underactuated simple mechanical systems to mechanical systems requiring second-order averaging. In this setting, we showed how the averaged system can be stabilized through an appropriate choice of error signals that are constant over each input cycle. A simulation demonstrated the method's utility. Ongoing work has applied these results to a fish-like robotic system and the snakeboard. In further developments we seek to generalize the averaging and approximate inversion results for mechanical systems to any order of approximation.

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